

# A NOTE ON THE FOUNDATIONS OF MECHANICS

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**ABSTRACT.** This short note is intended to review the foundations of mechanics, trying to present them with the greatest mathematical and conceptual clarity. It was attempted to remove most of inessential, even parasitic issues, which can hide the true nature of basic principles. The pursuit of that goal results in an improved understanding of some topics such as constrained systems, the nature of time or the relativistic forces. The Schrödinger and Klein-Gordon equations appear as conditions fulfilled by certain types of classical solutions of the field equations although the meaning of quantum equations is not, even remotely, exhausted by these cases. A part of this note comes from previous works [1, 2, 3].

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## 0. NOTATION AND REVIEW ON THE PRINCIPLES OF LAGRANGIAN MECHANICS

Let  $M$  be a smooth manifold,  $TM$  its tangent bundle. A vector tangent to  $TM$  or a vector field tangent to  $TM$  are said to be *vertical* when annihilate the subring  $\mathcal{C}^\infty(M)$  of  $\mathcal{C}^\infty(TM)$ . An 1-form on  $TM$  is said to be *horizontal* when it is incident with all of the vertical vectors. In local coordinates  $(x^1, \dots, x^n, \dot{x}^1, \dots, \dot{x}^n)$  on  $TM$ , the vertical vectors or fields are combinations of  $\partial/\partial \dot{x}^i$ , and the horizontal forms are of  $dx^i$ . A horizontal 1-form  $\alpha$  defines a function  $\dot{\alpha}$  on  $TM$ , by the rule  $\dot{\alpha}(v_a) = \langle \alpha, v_a \rangle$ , for each  $v_a \in TM$ . In particular, for each function  $f \in \mathcal{C}^\infty(M)$ ,  $\dot{f}$  is denoted simply by  $\dot{f}$ . The application

$$\dot{d}: \mathcal{C}^\infty(M) \rightarrow \mathcal{C}^\infty(TM), \quad f \mapsto \dot{d}(f) = \dot{f}$$

is, essentially, the differential.

A tangent field  $D$  on  $TM$  such that, as a derivation  $\mathcal{C}^\infty(M) \rightarrow \mathcal{C}^\infty(TM)$ , coincides with  $\dot{d}$  is a *second order differential equation*; for each  $v_a \in T_a M$  is, then,  $\pi_*(D_{v_a}) = v_a$ .

Since two second order differential equations act identically on  $\mathcal{C}^\infty(M)$ , they differ by a vertical field. For this reason, second order differential equations are the sections of an affine fiber bundle on  $TM$ , modeled on the vector bundle of vertical fields. From now on, a vertical field will be called a *force*; a vertical tangent vector  $V_{w_a} \in T_{w_a} TM$ , will be called a *force at*

$w_a$ . The value of a second order differential equation at a point  $w_a \in TM$  will be called *an acceleration* (at the velocity  $w_a$ ).

The vector structure of the fibers of  $TM$ , identifies canonically each force  $V_{w_a} \in T_{w_a}T_aM$  with a vector  $v_a$  of the fiber  $T_aM$ ; this vector  $v_a \in T_aM$  will be called the *geometric representative of the force*  $V_{w_a}$ .

Let  $T^*M$  be the cotangent bundle to  $M$ . The *Liouville form*  $\theta$  on  $T^*M$  is defined by  $\theta_{\alpha_a} = \alpha_a$  at each  $\alpha_a \in T^*M$ . Its exterior differential  $\omega_2 = d\theta$  is the *symplectic form* on  $T^*M$ . In local coordinates  $(x^1, \dots, x^n, p_1, \dots, p_n)$  of  $T^*M$  we have  $\theta = p_1 dx^1, \omega_2 = d\theta = dp_i \wedge dx^i$ .

Let  $T_2$  be a pseudoriemannian (= non degenerate) metric on  $M$ ;  $T_2$  establishes an isomorphism  $TM \simeq T^*M$ , by assigning to each vector  $v_a \in TM$  the 1-form  $\alpha_a = i_{v_a}T_2$ ; we will say that  $v_a$  is the *gradient* of  $\alpha_a$ . When we let the vector  $v_a$  to vary along  $TM$  we obtain  $i_dT_2 = \theta$ : the tautological field  $\dot{d}$  (which is the identity  $TM \rightarrow TM$ ) corresponds with the tautological form  $\theta$ .

The isomorphism  $TM \simeq T^*M$  produced by the metric  $T_2$  allow us to transport the structures from each bundle to the other. For this reason, we talk about the Liouville and the symplectic forms on  $TM$ , by using the same notation. The function  $T = \frac{1}{2} \dot{\theta}$  is the *kinetic energy* of  $(M, T_2)$ .

The whole of the Mechanics rests on the following

**Lema 0.** The metric  $T_2$  establishes a biunivocal correspondence between second order differential equations  $D$  and horizontal 1-forms  $\alpha$  by means of the equation

$$(0.1) \quad i_D\omega_2 + dT + \alpha = 0.$$

The tangent fields  $u$  on  $M$  which are intermediate integrals of  $D$  are those holding

$$(0.2) \quad i_u d(i_u T_2) + dT(u) + u^*\alpha = 0,$$

where  $u^*\alpha$  is the pull-back of  $\alpha$  by means of  $u: M \rightarrow TM$  and  $T(u)$  is the value of the function  $T$  when specialized to  $u$ .

A *classical mechanical system* is a manifold  $M$  (the *space of configuration*) endowed with a pseudoriemannian metric  $T_2$  and a horizontal form  $\alpha$ , the *work form*. The second order differential equation  $D$  which corresponds with  $\alpha$  according (0.1) is the *differential equation of the motion* and (0.1) is the *Newton equation* of the system  $(M, T_2, \alpha)$ .

The tangent field which corresponds with  $\alpha = 0$  is called *geodesic field*; it will be denoted by  $D_G$ . For an arbitrary system  $(M, T_2, \alpha)$ , the difference  $D - D_G$  is a vertical field, the *force* of the system. The geometric representative of the force is a field on  $TM$  with values in  $TM$ , which will be denoted by  $D^\nabla$  and called *covariant value* of  $D$ ; it holds  $i_{D^\nabla}T_2 + \alpha = 0$ , or, what is the same  $D^\nabla = -\text{grad } \alpha$ , and that is the general form of the Newton's law '*force=mass·acceleration*', by observing that, on each curve solution of  $D$  on  $M$  it holds  $D^\nabla = u^\nabla u$ , when  $u$  is the tangent vector along such a curve.

A system of *constraints* on  $M$  is a Pfaff system  $\Lambda$  on  $TM$  comprised by horizontal forms. The Newton equation (0.1) for a system with constraints  $(M, T_2, \alpha, \Lambda)$  is substituted by a congruence

$$(0.1') \quad i_D \omega_2 + dT + \alpha \equiv 0 \pmod{\Lambda}$$

joint with the *Virtual Works Principle*: the trajectories of the system remain into the subset of  $TM$  defined by  $\dot{\beta} = 0, \forall \beta \in \Lambda$ .

The determination of the field  $D$  by condition (0.1') plus the virtual works principle is possible under suitable conditions on the metric and the constraints; for example, when the metric is positive definite and  $\Lambda$  is the extension to  $TM$  of a regular Pfaff system on  $M$ . We will not dwell further on this issue, which now is not of direct interest for our purposes.

## 1. CONSERVATIVE SYSTEMS

When  $\alpha$  is an exact differential form, the system  $(M, T_2, \alpha)$  is said to be *conservative*; since  $\alpha$  is horizontal, the potential function  $V$  of which  $\alpha$  is its differential, must belong to the subring  $\mathcal{C}^\infty(M)$  of  $\mathcal{C}^\infty(TM)$ . The sum  $H = T + V$  is called the *hamiltonian* of the system and the Newton equation (0.1) is, in this case,

$$(1.1) \quad i_D \omega_2 + dH = 0.$$

When written in coordinates  $(x^i, p_i)$  of  $T^*M$ , (1.1) is the system of *Hamilton canonical equations*. From (1.1) and the classical argument based on Stokes theorem, it follows the *Maupertuis principle*: the trajectories of the mechanical system on  $M$  are extremal for  $\int \theta$  with fixed end points and  $H = \text{const.}$

The equation (0.2) for the intermediate integrals becomes in this case:

$$(1.2) \quad i_u d(i_u T_2) + dH(u) = 0$$

In particular, the fields  $u$  which, as submanifolds of  $TM$ , are lagrangian, hold  $d(i_u T_2) = d\theta|_u = 0$ , so that

$$(1.3) \quad dH(u) = 0 \quad \text{or} \quad H(u) = \text{const.}$$

This is the *Hamilton-Jacobi equation*.

Let us demand somewhat more of  $u$ : to be, in addition to lagrangian, conservative, that is to say, of null divergence. Since  $u$  is lagrangian, locally we have  $u = \text{grad } f$ , for a suitable function  $f$  in  $M$ ; then,  $\text{div } u = \Delta f = 0$ .

We have a formula, which holds for  $T_2$  without conditions,

$$(1.4) \quad \Delta e^{i\phi} = e^{i\phi} (i\Delta\phi - T^2(d\phi, d\phi)),$$

for all local function  $\phi$  on  $M$ . By applying it in our case to the harmonic function  $\phi = f/h$  ( $h$  is a constant) and calling  $\Psi := e^{if/h}$  we get

$$\Delta \Psi = -\frac{\Psi}{h^2} T^2(df, df) = -\frac{\Psi}{h^2} T_2(u, u) = -\frac{2\Psi}{h^2} (H(u) - V),$$

and then, if  $E$  is the constant value of  $H(u)$ :

$$(1.5) \quad \left( \frac{h^2}{2} \Delta - V \right) \Psi = -E\Psi$$

This is the *Schrödinger equation*. Conversely, if (1.5) holds  $\|\Psi\| = 1$  ( $E$  real), then we get an intermediate integral which is lagrangian and conservative. In addition, by taking the real or the imaginary parts of these type of fields  $\Psi$ , we get solutions which are not interpretable in classical terms.

## 2. LENGHT AND TIME

In Mechanics, *time* is the parameter of each curve-solution of field  $D$  which govern the evolution of the system. However, it cannot be a function on  $TM$  which serves as a parameter for all the solutions of all the second order differential equations on  $M$ ; on  $TM$  does not exist any function which can be called time. The natural object which serves to parametrice all the curve-solutions of all the second order differential equations is a class of 1-forms, the *class of time*, comprised by the local horizontal 1-forms  $\alpha$  such that  $\dot{\alpha} = 1$ ; for each horizontal 1-form  $\beta$ , in the open subset where  $\dot{\beta} \neq 0$ , the quotient  $\beta/\dot{\beta}$  belongs to the class of time. Two forms belonging to the class of time differ, in their common domain, by a form of the contact system  $\Omega$  of  $TM$ ; the class of time is that of  $\theta/\dot{\theta} \bmod \Omega$ .

We can *choose a time* for  $(M, T_2)$  by giving a horizontal form  $\tau$  on  $TM$  and considering as possible trajectories just those which are contained in the subset  $\dot{\tau} = 1$  of  $TM$ ; with this, the space of states is restricted to that subset. Assuming that  $\dot{\tau} = 1$  or, more generally,  $\dot{\tau} = \text{const.}$  defines a submanifold of  $TM$ , for each mechanical system  $(M, T_2, \alpha)$ , the field  $D$  governing the evolution prior to the time constraint, must be modified to another field  $\overline{D}$  holding the congruence

$$(0.1'') \quad i_{\overline{D}}\omega_2 + dT + \alpha \equiv 0 \pmod{\tau}$$

and, in substitution of the Virtual Works Principle, the condition:  $\overline{D}$  is tangent to the manifolds  $\dot{\tau} = \text{const.}$

For instance, when  $\tau$  is a form on  $M$ , we have

$$(2.1) \quad \overline{D} = D - \frac{1}{\|\tau\|^2} \left( \langle \tau, D^\nabla \rangle + \Pi_{\text{grad } \tau}(\dot{d}, \dot{d}) \right) \text{Grad } \tau$$

out of the set where  $\|\tau\| = 0$ ; that set is void in the most classical case:  $\tau$  without zeroes and  $T_2$  positive definite. In (2.1),  $\text{Grad } \tau$  is the vertical field which corresponds to  $\text{grad } \tau$  and  $\Pi_u$  is the second fundamental form of the tangent field  $u$  with respect to  $T_2$ .

The form  $\frac{\theta}{\|\theta\|} = \frac{\theta}{\sqrt{|\theta|}}$  is defined on the open set of  $TM$  where  $\dot{\theta} (= 2T) \neq 0$ ; when  $T_2$  is positive definite, that open is  $TM$  without the zero section; when  $T_2$  has the signature of Minkowski, the open set excludes the ‘light cones’, etc.

In its domain  $\frac{\theta}{\|\theta\|}$  is invariant under the group of homoteties on fibers; in fact, under the whole of groups generated by fields  $\mu V$ , where  $V$  is the infinitesimal generator of homoteties and  $\mu \in \mathcal{C}^\infty(TM)$ . For this reason,  $\frac{\theta}{\|\theta\|}$  projects to the open of the space of 1-jets of curves  $J_1^1 M$  which is image of  $\dot{\theta} \neq 0$ .

Physicists call the length *proper time*; this is why we can call  $\frac{\theta}{\sqrt{|\dot{\theta}|}}$  the *length form* or the *proper time form*.

A non-parametrized curve  $\gamma$  in  $M$ , defines canonically a curve  $\tilde{\gamma}$  in  $J_1^1 M$ ; the integral of  $\frac{\theta}{\|\theta\|}$  along  $\tilde{\gamma}$  is the *length* of  $\gamma$ , which is independent of any parametrization: it is the fact of  $\frac{\theta}{\|\theta\|}$  being projectable by  $TM \rightarrow J_1^1 M$  what enables us to define the length or proper time. Except for some technical detail, similar reasons allow us to define the volume of  $r$ -dimensional submanifolds of  $M$  ( $1 \leq r \leq n$ ).

### 3. RELATIVISTIC FORCES

We have considered the forms  $\theta$ ,  $\frac{\theta}{\dot{\theta}}$ ,  $\frac{\theta}{\|\theta\|}$ , the last two defined on the open  $\dot{\theta} \neq 0$ . The form  $\frac{\theta}{\dot{\theta}}$  and all those which are congruent with it mod  $\Omega$ , specialize on each curve solution of each second order differential equation  $D$ , as the differential of the natural parameter, let us say  $dt$ . The form  $\frac{\theta}{\|\theta\|} = \frac{\theta}{2T}$  specializes on each curve of  $TM$  as the length form or length element  $ds$ ; its integral along such a curve is the length of the curve once projected to  $M$ . Physicists parametrize the “relativistic” motions of particles by the “proper time” of each particle. Since in a mechanical system, without any condition on the configuration space  $M$  or the metric  $T_2$ , the unique possible parameter for the trajectories (of second order differential equations) is the class of time, it follows that on the relativistic trajectories must be suitable as a natural parameter, both  $\frac{\theta}{\dot{\theta}}$  and  $\frac{\theta}{\|\theta\|}$ . That condition determines uniquely if a trajectory is relativistic or not: a curve solution of a second order differential equation on  $TM$  is “relativistic” if on it holds  $\dot{\theta} = 1$ ,  $\dot{\theta} = -1$  or  $\dot{\theta} = 0$ .

This is why, provisionally, it is natural to distinguish between relativistic or no relativistic mechanical systems according to the following criterium:  $(M, T_2, \alpha)$  is relativistic if the field  $D$  which governs the evolution of such a system is tangent to the hypersurfaces  $\dot{\theta} = 1$ ,  $\dot{\theta} = -1$ ,  $\dot{\theta} = 0$  of  $TM$ .

Let us decompose the field  $D$  as a sum  $D = D_G + W$ , where  $D_G$  is the geodesic field and  $W$  is a vertical field.  $D_G$  is relativistic; its Newton equation is  $i_{D_G} \omega_2 + dT = 0$ , and then  $D_G \dot{\theta} = D_G(2T) = 0$ . Hence,  $D$  is tangent to the hypersurfaces  $\dot{\theta} = 0, \pm 1$ , if and only if  $W$  is. Now, since  $W$  is vertical, we have  $W \dot{\theta} = \langle w, \theta \rangle$ , where  $w$  is the geometric representative of  $W$  ( $w = D^\nabla$ ). In local coordinates,  $\langle w, \theta \rangle = g_{ij} w^i \dot{x}^j$ ; if the functions  $w^i$  are homogeneous (of arbitrary degree) on the  $\dot{x}$ ,  $\langle w, \theta \rangle$  is homogeneous in the  $\dot{x}$ , hence, if vanishes on  $\dot{\theta} = 1$  then vanishes on each  $\dot{\theta} = \text{const}$ .

As a consequence, the relativistic fields  $D = D_G + W$  when  $W$  is an homogeneous of the velocities  $\dot{x}$ , are tangent to all of the submanifolds  $\dot{\theta} = \text{const}$ ; this is to say,  $\dot{\theta}$  must be a first integral of  $D$ , hence, also of  $W$ ; from the equation (0.1) follows  $DT + \dot{\alpha} = 0$ , and then  $DT = 0 \Leftrightarrow \dot{\alpha} = 0 \Leftrightarrow WT = 0$ .

Summing up: it is natural to adopt the following

**Definition 3.1.** The mechanical system  $(M, T_2, \alpha)$  is relativistic if the corresponding field  $D$  holds  $D\dot{\theta} = 0$ . This condition is equivalent to the field of forces  $W = D - D_G$  holding  $W\dot{\theta} = 0$ .

And it holds the

**Theorem 3.2.**  $(M, T_2, \alpha)$  is relativistic if and only if the work form  $\alpha$  belongs to the contact system  $\Omega$  of  $TM$ .

This condition is independent of the metric!

Since the contact system does not contain exact forms, except the 0 form, there is no other relativistic and conservative system but the geodesic one. The fact that the conservative systems (except the geodesic) never are relativistic, has nothing to do with the absence of action at a distance or the finiteness of the speed of light, because it is an exclusive consequence of the lack of closed forms in the contact system.

In local coordinates, a system of generators of  $\Omega$  is the comprised by forms  $\dot{x}^h dx^j - \dot{x}^j dx^h = i_{\dot{d}}(dx^h \wedge dx^j)$ . It follows that for every 1-form  $\alpha$  of the contact system  $\Omega$  there exists a horizontal hemisymmetric tensor field (a horizontal 2-form)  $F_2$  on  $TM$  such that  $i_{\dot{d}}F_2 = \alpha$ : every relativistic force is “produced” by an hemisymmetric covariant tensor field, of order 2, horizontal on  $TM$ ; in general, the field  $F_2$  is not completely determined by  $\alpha$ . When, the force field  $W$  depends linearly on the velocities (that is to say, of the  $\dot{x}$ ’s), also is this way for  $\alpha = -i_w T_2$ ; in such a case, there exists a unique 2-form  $F_2$  on  $M$  such that  $i_{\dot{d}}F_2 = \alpha$ ; the 2-forms on  $M$  are the same objects as the contact 1-forms on  $TM$  which linearly depend on the  $\dot{x}$ . We have

**Theorem 3.3.** *The relativistic forces which depend linearly on the velocities correspond biunivocally with 2-forms on  $M$ . To the 2-form  $F_2$  it corresponds the force 1-form  $\alpha = i_{\dot{d}}F_2$ .*

#### 4. ELECTROMAGNETIC FIELDS

By an extension of the classical case, we will call *electromagnetic field* on  $M$  to each closed 2-form  $F_2$  on  $M$ . The 1-form  $\alpha = i_{\dot{d}}F_2$ , which is the translation of  $F_2$ , will be called *Lorentz force form* of the electromagnetic field. This form does not depend on the metric on  $M$ ; once the metric is fixed, the Newton equation

$$(0.1'') \quad i_D \omega_2 + dT + i_{\dot{d}} F_2 = 0$$

determines the second order differential equation  $D$ , to which we can call the *Lorentz field* produced by the electromagnetic field  $F_2$  on  $(M, T_2)$ ; the associated force  $D - D_G$  is the *Lorentz force*.

Looking at the Newton equation (0.1''), it is natural to consider the following 2-form on  $TM$ :

$$(4.1) \quad \omega_F = \omega_2 + F_2.$$

$\omega_F$  is a new symplectic form on  $TM$ ; the Newton equation (0.1'') is written:

$$(4.2) \quad i_D \omega_F + dT = 0$$

Thus, the Lorentz field  $D$  is the hamiltonian field which corresponds to the hamiltonian  $T$  in the symplectic structure  $\omega_F$ .

The equation (0.2) is, for the intermediate integrals of the Lorentz field:

$$(4.3) \quad i_u(F_2 + di_u T_2) + dT(u) = 0$$

or

$$(4.3') \quad i_u(\omega_F|_u) + dT(u) = 0$$

where  $\omega_F|_u$  is the specialization of  $\omega_F$  to the section  $u$  of  $TM$ .

The intermediate integrals of the Lorentz field which are lagrangian for the symplectic form  $\omega_F$  hold

$$(4.3'') \quad \omega_F|_u = 0$$

hence

$$(4.4) \quad T(u) = \text{const.}$$

which is the Hamilton-Jacobi equation for the  $\omega_F$ -lagrangian intermediate integrals of  $D$ .

The closed 2-form  $F_2$  is locally exact; locally  $F_2$  can be written as  $F_2 = d(i_A T_2)$ , where the field  $A$  is the *potential vector* for  $F_2$ . The fact that a tangent field  $u$  on  $M$  is a section a  $\omega_F$ -lagrangian section of  $TM$  is written as

$$d(i_{u+A} T_2) = 0,$$

so that

$$(4.5) \quad i_{u+A} T_2 = df$$

locally, with  $f \in C^\infty(M)$ .

If the field  $u + A$  is conservative ( $\text{div}(u + A) = 0$ ), we have  $\delta df = \Delta f = 0$  and the formula (1.4) applied on  $\Psi = e^{i\frac{f}{h}}$  ( $h$  is a constant) gives us

$$\begin{aligned}
\Delta \Psi &= -\frac{\Psi}{h^2} T^2(df, df) \\
&= -\frac{\Psi}{h^2} T_2(u + A, u + A) \\
&= -\frac{\Psi}{h^2} (\|u\|^2 + \|A\|^2 + 2T_2(A, u)) \\
&= -\frac{\Psi}{h^2} (\|u\|^2 + \|A\|^2 + 2\langle A, df \rangle - 2T_2(A, A)) \\
&= -\frac{\Psi}{h^2} (\|u\|^2 - \|A\|^2 + 2A(f)) \\
&= -\frac{1}{h^2} (\|u\|^2 - \|A\|^2 - 2ihA) \Psi.
\end{aligned}$$

Equation (4.4) gives us a constant  $\|u\|^2 = m^2$ ; then, it follows

$$(4.6) \quad \left( \Delta - 2\frac{i}{h}A + \frac{1}{h^2}(m^2 - \|A\|^2) \right) \Psi = 0$$

which is the Klein-Gordon equation:

**Theorem 4.1.** *The Klein-Gordon equation (4.6) characterizes the tangent fields  $u$  on  $M$  which are intermediate integral of the Lorentz field  $D$ , lagrangian with respect the symplectic structure  $\omega_F = \omega_2 + F_2$  and such that  $\text{div}(u + A) = 0$ , where  $A$  is a vector potential for  $F_2$ .*

**Remark on the first pair of Maxwell equations** The first pair of Maxwell equations  $\delta F_2 = J^*$  is interpretable as a *definition* of electric current. However, it is not true, in the classical case, that the “electric fluid” has necessarily as pathlines the curves solution of the field  $J$ , because we can change  $F_2$ , and with it the Lorentz force, without any change of  $J$ . In the classical case,  $M = \mathbb{R}^4$ -Minkowski, we can choose the vector-potential  $A$  in such a way that it holds the “Lorentz gauge” condition,  $\text{div} A = 0$  and, then, (4.6) is a condition on the intermediate integral  $u$ :  $u$  is lagrangian and  $\text{div} u = 0$ . If we assume that  $J$  is a particle flow which obey to the Lorentz force,  $J = u$  is an intermediate integral of that force and automatically, by its very definition  $J^* = \delta F_2$ , it holds  $\text{div} J = 0$  and the Klein-Gordon equation (4.6) is the condition to the current  $J$  be lagrangian.

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